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REALIZATION PROBLEM FOR POSITIVE FRACTIONAL HYBRID 2D LINEAR SYSTEMS *

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Abstract

The realization problem for positive fractional hybrid 2D linear systems is addressed. A method for computation of positive fractional realizations of a given proper 2D transfer matrix is proposed. Sufficient conditions for the existence of positive fractional realizations of a proper 2D transfer matrix are established. A procedure for computation of positive fractional realizations is proposed and illustrated by a numerical example.

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Key Words and Phrases: fractional hybrid linear system, 2D system, positive system, computation, existence, realization, transfer matrix

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

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Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [2], [3] and in [29]. The realization problem for positive discrete-time and continuous-time systems without and with delays was considered in [1], [2], [3]-[7], [12], [15], and for positive fractional 2D hybrid systems in [14].

The realization problem arises in many applications when knowing the transfer function (obtained as a result of the identification procedure) we are looking for the state space description of the system.

The reachability, controllability and minimum energy control of positive linear discrete-time systems with delays have been considered in [16], [8].

The relative controllability of stationary hybrid systems has been investigated in [19] and the observability of linear differential-algebraic systems with delays has been considered in [20]. A new class of positive 2D hybrid linear system has been introduced in [11], and of positive fractional hybrid 2D systems in [13]. The new class of positive 2D hybrid systems is described by two vector equations in two independent variables, continuous and discrete ones. The first equation is a differential one and the second one is a difference equation.

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19^{th} century, see [21], [17], [23], [24]. This idea has been used by engineers for modelling different processes in the late 1960s, as for example in [28], [30]. Mathematical fundamentals of fractional calculus are given in the monographs [17], [21], [26], [23], [24]. A generalization of the Kalman filter for fractional order systems has been proposed in [27]. Some other applications of fractional order systems can be found in [23], [21], [28], [30]. Positive fractional continuous-time linear systems have been introduced in [10] and cone fractional systems in [9].

In this paper a method for computation of positive fractional realizations of a given proper 2D transfer matrix of hybrid linear system is proposed.

The paper is organized as follows. In Section 2 the solutions of state equations of 2D linear hybrid systems and the necessary and sufficient conditions for the positivity of the hybrid systems are recalled. The realization problem for positive fractional 2D hybrid linear systems is formulated in Section 3. The solution of the problem for the m-inputs and p-outputs positive fractional systems is presented in Section 4. Sufficient conditions for the existence of positive fractional realizations of a proper 2D transfer matrix are established and a procedure for computation of the positive fractional realizations is proposed. An example illustrating the procedure is presented in Section 5. Concluding remarks are given in Section 6.

To the best knowledge of the author, the realization problem for positive fractional 2D hybrid linear systems has not been considered yet.

2. Positive fractional 2D hybrid systems

Let $\Re^{n\times m}$ be the set of $n\times m$ real matrices. The set of $n\times m$ matrices with nonnegative entries will be denoted by $\Re_+^{n\times m}$, and $\Re_+^n := \Re_+^{n\times 1}$. Let Z_+ be the set of nonnegative integers. The $n\times n$ identity matrix will be denoted by I_n.

Consider the hybrid fractional 2D system

$$\frac{d^\alpha x_1(t,i)}{dt^\alpha} = A_{11}x_1(t,i) + A_{12}x_2(t,i) + B_1u(t,i), \quad t\in\Re_+ = [0,+\infty], \tag{1a}$$

$$\triangle^\beta x_2(t,i+1) = A_{21}x_1(t,i) + A_{22}x_2(t,i) + B_2u(t,i), \quad i\in Z_+, \tag{1b}$$

$$y(t,i) = C_1x_1(t,i) + C_2x_2(t,i) + Du(t,i), \tag{1c}$$

where α $(0<\alpha<1)$ is the order of the fractional derivative, β $(0<\beta<1)$ is the order of the fractional difference, $x_1(t,i)\in\Re^{n_1}$, $x_2(t,i)\in\Re^{n_2}$, $u(t,i)\in\Re^m$, $y(t,i)\in\Re^p$ and A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D are real matrices with appropriate dimensions. The boundary conditions for (1a) and (1b) have the form

$$x_1(0,i) = x_1(i), \quad i\in Z_+ \text{ and } x_2(t,0) = x_2(t), \quad t\in\Re_+\,. \tag{2}$$

Note that the fractional 2D hybrid system (1) has a similar structure as the Roesser model [25], [3]. The following Caputo definition of the fractional derivative [24], [17]

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau \quad \left(x^{(n)}(\tau) = \frac{d^n x(\tau)}{d\tau^n}\right) \tag{3}$$
$$n-1<\alpha<n\in N=\{1,2,...\}$$

will be used, where

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt \tag{4}$$

is the Gamma function.

The fractional difference of the order β of x_i is defined by

$$\triangle^{\beta} x_i = \sum_{k=0}^{i} (-1)^k \left(\begin{array}{c} \beta \\ k \end{array} \right) x_{i-k}, \quad 0 < \beta < 1, \ i \in Z_+, \tag{5}$$

where

$$\left(\begin{array}{c} \beta \\ k \end{array} \right) = \begin{cases} 1 \ \text{ for } \ k = 0 \\ \dfrac{\beta(\beta-1)...(\beta-k+1)}{k!} \ \text{ for } \ k = 1, 2, ... \end{cases} \tag{6}$$

Using (5), we may write the equation (1b) in the form

$$x_2(t, i+1) = A_{21} x_1(t, i) + A_{22} x_2(t, i) + \sum_{k=1}^{i+1} c_k x_2(t, i-k+1) + B_2 u(t, i), \ i \in Z_+, \tag{7}$$

where

$$c_k = c_k(\beta) = (-1)^{k-1} \binom{\beta}{k}, \ k = 1, 2, ... \tag{8}$$

Remark. From (6) and (8) it follows that the coefficients c_k strongly decrease when k increases. Therefore, in practical problems it is assumed that i is bounded by a natural number L and $\sum_{k=1}^{i+1} c_k x_2(t, i-k+1) = \sum_{k=1}^{L+1} c_k x_2(t, i-k+1)$.

Theorem 1. ([13]) *The solutions to the equations (1a) and (7) with the boundary conditions (2) have the forms*

$$\begin{aligned} \left[\begin{array}{c} x_1(t,i) \\ x_2(t,i) \end{array} \right] &= \sum_{p=0}^{\infty} \sum_{q=0}^{i} \frac{T_{p,i-q} B_{10}}{\Gamma[(p+1)\alpha]} \int_0^t (t-\tau)^{(p+1)\alpha-1} u(\tau, q) d\tau \\ &+ \sum_{p=0}^{\infty} \sum_{q=0}^{i} \frac{T_{p,i-q-1} B_{01}}{\Gamma(\alpha p + 1)} \int_0^t (t-\tau)^{\alpha p - 1} u(\tau, q) d\tau \\ &+ \sum_{p=0}^{\infty} \sum_{q=0}^{i} \frac{T_{p,i-q}}{\Gamma(\alpha p+1)} (t-\tau)^{p\alpha} \begin{bmatrix} x_1(0,q) \\ 0 \end{bmatrix} + \sum_{p=0}^{\infty} \frac{T_{p,i}}{\Gamma(\alpha p)} \int_0^t (t-\tau)^{p\alpha} \begin{bmatrix} 0 \\ x_2(t,0) \end{bmatrix} d\tau, \end{aligned} \tag{9}$$

where

$$T_{pq} = \begin{cases} I_{n_1+n_2} \ \ \textit{for} \ \ p = q = 0 \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \ \ \textit{for} \ \ p = 1, q = 0 \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} + I_{n_2} c_1 \end{bmatrix} \ \ \textit{for} \ \ p = 0, q = 1 \\ T_{10} T_{p-1,q} + T_{01} T_{p,q-1} \ \ \textit{for} \ \ p + q > 0 \\ 0 \ \ \textit{for} \ \ p < 0 \ \ \textit{or/and} \ \ q < 0, \end{cases} \tag{10a}$$

$$B_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix},$$

and

$$T_{0q} = (T_{01})^q + \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} c_q \end{bmatrix}, \ q = 2, 3, ... \tag{10b}$$

Lemma 1. ([13]) *If* $0 < \beta < 1$, *then*

$$c_k = c_k(\beta) > 0 \ \textit{for} \ k = 1, 2, ... \tag{11}$$

Lemma 2. ([13]) *If* $0 < \beta < 1$ *and* $A_{21} \in \Re_+^{n_2 \times n_1}$, $A_{22} \in \Re_+^{n_2 \times n_2}$, *then*

$$T_{0q} \in \Re_+^{(n_1+n_2)\times(n_1+n_2)} \ \ \textit{for} \ q = 2, 3, ... \tag{12}$$

Definition 1. ([13]) The fractional 2D hybrid system (1) is called (internally) positive, if and only if $x_1(t, i) \in \Re_+^{n_1}$, $x_2(t, i) \in \Re_+^{n_2}$, $t \geq 0$, $i \in Z_+$ for any boundary conditions

$$x_1(0, i) = x_1(i) \in \Re_+^{n_1}, \ i \in Z_+ \text{ and } x_2(t, 0) = x_2(t) \in \Re_+^{n_2}, \ t \geq 0 \tag{13}$$

and all inputs $u(t, i) \in \Re_+^m$, $t \geq 0$, $i \in Z_+$.

Let M_n be the set of $n \times n$ Metzler matrices (real matrices with non-negative off-diagonal entries).

Theorem 2. ([13]) *The fractional 2D hybrid system (1) is (internally) positive, if and only if*

$$\begin{array}{l} A_{11} \in M_{n_1}, \quad A_{12} \in \Re_+^{n_1 \times n_2}, \quad A_{21} \in \Re_+^{n_2 \times n_1}, \quad A_{22} \in \Re_+^{n_2 \times n_2} \\ B_1 \in \Re_+^{n_1 \times m}, \quad B_2 \in \Re_+^{n_2 \times m}, \quad C_1 \in \Re_+^{p \times n_1}, \quad C_2 \in \Re_+^{p \times n_2} \quad D \in \Re_+^{p \times m}. \end{array} \tag{14}$$

3. Problem formulation

Using the Laplace transform with respect to t and Z transform with respect to i to the equations (1) it is easy to show that the transfer matrix of the system has the form

$$T(s,z) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} I_{n_1}s^{\alpha} - A_{11} & -A_{12} \\ -A_{12} & I_{n_2}(z-c) - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D\,, \tag{15}$$

where

$$c = \sum_{k=1}^{1+1} c_k z^{1-k} = \sum_{k=1}^{i+1} (-1)^{k-1} \binom{\beta}{k} z^{1-k}. \tag{16}$$

The transfer matrix $T(s,z)$ is called proper, if and only if

$$\lim_{s\to\infty,\ z\to\infty} T(s,z) = K \in \Re^{p\times m}$$

and it is called strictly proper, if and only if $K = 0$.

The transfer matrix (15) is proper since

$$\lim_{s\to\infty,\ z\to\infty} \begin{bmatrix} I_{n_1}s^{\alpha} - A_{11} & -A_{12} \\ -A_{12} & I_{n_2}(z-c) - A_{22} \end{bmatrix} = 0$$

and

$$\lim_{s\to\infty,\ z\to\infty} T(s,z) = D. \tag{17}$$

Definition 2. Matrices (14) are called a positive fractional realization of a given transfer matrix $T(s,z)$, if they satisfy the equality (15). A realization is called minimal, if the dimension of A_{kl} $(k,l = 1,2)$ is minimal among all realizations of $T(s,z)$.

The positive fractional realization problem of the 2D hybrid linear systems can be stated as follows. Given a proper transfer matrix $T(s,z)$, find its positive fractional realization (14).

In this paper sufficient conditions for the existence of the positive fractional realization (14) of a given proper transfer matrix $T(s,z)$ are established and a procedure for computation of the realization is proposed.

4. Solution of the problem

From (15) it follows that the transfer matrix of fractional 2D hybrid system (1) is a proper rational matrix in s^{α} and $(z-c)$.

The proposed method of solving of the realization problem for positive fractional hybrid 2D systems is based on the following two lemmas.

LEMMA 3. *The transfer matrix (15) can be written in the form*

$$T(s,z) = C(s)[I_{n_2}(z-c) - A(s)]^{-1}B(s) + D(s), \tag{18a}$$

or

$$T(s,z) = C(z)[I_{n_1}s^{\alpha} - A(z)]^{-1}B(z) + D(z), \tag{18b}$$

where

$$\begin{aligned} A(s) &= A_{22} + A_{21}[I_{n_1}s^{\alpha} - A_{11}]^{-1}A_{12}, \quad B(s) = A_{21}[I_{n_1}s^{\alpha} - A_{11}]^{-1}B_1 + B_2,\\ C(s) &= C_2 + C_1[I_{n_1}s^{\alpha} - A_{11}]^{-1}A_{12}, \quad D(s) = C_1[I_{n_1}s^{\alpha} - A_{11}]^{-1}B_1 + D \end{aligned} \tag{19a}$$

and

$$\begin{aligned} A(z) &= A_{11} + A_{12}[I_{n_2}(z-c) - A_{22}]^{-1}A_{21},\\ B(z) &= A_{12}[I_{n_2}(z-c) - A_{22}]^{-1}B_2 + B_1,\\ C(z) &= C_1 + C_2[I_{n_2}(z-c) - A_{22}]^{-1}A_{21},\\ D(z) &= C_2[I_{n_2}(z-c) - A_{22}]^{-1}B_2 + D. \end{aligned} \tag{19b}$$

P r o o f. Using the Laplace transform to t and the Z transform to i for the equations (1a) and (7) with zero boundary conditions (2) we obtain

$$s^{\alpha}X_1 = A_{11}X_1 + A_{12}X_2 + B_1U, \tag{20a}$$

$$(z-c)X_2 = A_{21}X_1 + A_{22}X_2 + B_2U, \tag{20b}$$

$$Y = C_1X_1 + C_2X_2 + DU, \tag{20c}$$

where

$$X_k = X_k(s,z) = Z\{L[x_k(t,i)]\},\ k = 1,2;$$
$$U = U(s,z) = Z\{L[u(t,i)]\},\ Y = Y(s,z) = Z\{L[y(t,i)]\},$$

and c is defined by (16).

From (20b) we have

$$X_2 = [I_{n_2}(z-c) - A_{22}]^{-1}(A_{21}X_1 + B_2U). \tag{21}$$

Substitution of (21) into (20a) yields

$$X_1 = [I_{n_1}s^{\alpha} - A_{11} - A_{12}[I_{n_2}(z-c) - A_{22}]^{-1}]^{-1}(A_{12}[I_{n_2}(z-c) - A_{22}]^{-1}B_2 + B_1)U \tag{22}$$

and after substituting (22) into (21) we obtain

$$\begin{aligned} X_2 = {} & [I_{n_2}(z-c) - A_{22}]^{-1}A_{21}\{[I_{n_1}s^{\alpha} - A_{11} - A_{12}\\ & \times [I_{n_2}(z-c) - A_{22}]^{-1}]^{-1}(A_{12}[I_{n_2}(z-c) - A_{22}]^{-1}B_2 + B_1)U\}\\ & + [I_{n_2}(z-c) - A_{22}]^{-1}B_2U. \end{aligned} \tag{23}$$

Finally substituting of (22) and (23) into (20c) yields (18b).

The proof of (18a) is similar (by interchanging the role of s^{α} and $(z-c)$). ∎

LEMMA 4. *Let $A(s)$, $B(s)$, $C(s)$, $D(s)$ and $A(z)$, $B(z)$, $C(z)$, $D(z)$ be defined by (19a) and (19b). Then*

$$\lim_{s\to\infty}\begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} = \begin{bmatrix} A_{22} & B_2 \\ C_2 & D \end{bmatrix} \tag{24a}$$

$$\lim_{z\to\infty}\begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} = \begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix} \tag{24b}$$

and

$$\lim_{s\to\infty} s\begin{bmatrix} A(s)-A_{22} & B(s)-B_2 \\ C(s)-C_2 & D(s)-D \end{bmatrix} = \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix}\begin{bmatrix} A_{12} & B_1 \end{bmatrix} \tag{25a}$$

$$\lim_{s\to\infty} z\begin{bmatrix} A(z)-A_{11} & B(z)-B_1 \\ C(z)-C_1 & D(z)-D \end{bmatrix} = \begin{bmatrix} A_{12} \\ C_2 \end{bmatrix}\begin{bmatrix} A_{21} & B_2 \end{bmatrix}. \tag{25b}$$

P r o o f. From (19a) we have

$$\begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} = \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix}[I_{n_1}s^{\alpha}-A_{11}]^{-1}\begin{bmatrix} A_{12} & B_1 \end{bmatrix} + \begin{bmatrix} A_{22} & B_2 \\ C_2 & D \end{bmatrix}. \tag{26}$$

Taking into account that $\lim\limits_{s\to\infty}[I_{n_1}s^{\alpha}-A_{11}]^{-1}=0$, from (26) we obtain (24a). The proof of (24b) is similar. From (26) we have

$$\lim_{s\to\infty} s\begin{bmatrix} A(s)-A_{22} & B(s)-B_2 \\ C(s)-C_2 & D(s)-D \end{bmatrix}$$

$$= \lim_{s\to\infty} s\begin{bmatrix} A_{21} \\ C_1 \end{bmatrix}[I_{n_1}s^{\alpha}-A_{11}]^{-1}\begin{bmatrix} A_{12} & B_1 \end{bmatrix} = \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix}\begin{bmatrix} A_{12} & B_1, \end{bmatrix}$$

since $\lim\limits_{s\to\infty} s^{\alpha}[I_{n_1}s-A_{11}]^{-1}=I_{n_1}$. The proof for (25b) is similar. ∎

It is well-known [3] that any 2D transfer matrix can be always written in the form

$$T(s,z)=\frac{N_{q_1}(z)}{d_{q_1}(z)}+\frac{N_{q_1-1}(z)s^{\alpha q_1-1}+\ldots+N_1(z)s^{\alpha}+N_0(z)}{s^{\alpha q_1}+d_{q_1-1}(z)s^{\alpha q_1-1}+\ldots+d_1(z)s^{\alpha}+d_0(z)}, \tag{27a}$$

or

$$T(s,z)=\frac{N_{q_2}(s)}{d_{q_2}(s)}+\frac{N_{q_2-1}(s)(z-c)^{q_2-1}+\ldots+N_1(s)(z-c)+N_0(s)}{(z-c)^{q_2}+d_{q_2-1}(s)(z-c)^{q_2-1}+\ldots+d_1(s)(z-c)+d_0(s)}, \tag{27b}$$

where

$$N_{q_1}(z) \in R^{p\times m}[z], \quad N_{q_2}(s) \in R^{p\times m}[s], \quad d_{q_1}(z) \in R[z], \quad d_{q_2}(s) \in R[s],$$

$$N_k(z) \in R^{p\times m}(z), \quad d_k(z) \in R(z), \quad k = 0, 1, ..., q_1 - 1$$

$$N_l(s) \in R^{p\times m}(s), \quad d_l(s) \in R(s), \quad l = 0, 1, ..., q_2 - 1,$$

and $R^{p\times m}[z]$ ($R^{p\times m}[s]$) is the set of $p \times m$ polynomial matrices in z (s).

Using one of the well-known methods [3], we can find a realization $A(z)$, $B(z)$, $C(z)$, $D(z)$ of (27a) and a realization $A(s)$, $B(s)$, $C(s)$, $D(s)$ of (27b). For example, it is easy to verify that the matrices

$$\begin{array}{l} A(z) = \begin{bmatrix} 0 & 0 & ... & 0 & -d_0(z)I_p \\ I_p & 0 & ... & 0 & -d_1(z)I_p \\ 0 & I_p & ... & 0 & -d_2(z)I_p \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & I_p & -d_{q_1-1}(z)I_p \end{bmatrix}, \quad B(z) = \begin{bmatrix} N_0(z) \\ N_1(z) \\ N_2(z) \\ \vdots \\ N_{q_1-1}(z) \end{bmatrix} \\ C(z) = \begin{bmatrix} 0 & 0 & ... & 0 & I_p \end{bmatrix}, \quad D(z) = \frac{N_{q_1}(z)}{d_{q_1}(z)} \end{array} \tag{28a}$$

are a realization of (27a), and the matrices

$$\begin{array}{l} A(s) = \begin{bmatrix} 0 & 0 & ... & 0 & -d_0(s)I_p \\ I_p & 0 & ... & 0 & -d_1(s)I_p \\ 0 & I_p & ... & 0 & -d_2(s)I_p \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & I_p & -d_{q_2-1}(s)I_p \end{bmatrix}, \quad B(s) = \begin{bmatrix} N_0(s) \\ N_1(s) \\ N_2(s) \\ \vdots \\ N_{q_2-1}(s) \end{bmatrix} \\ C(s) = \begin{bmatrix} 0 & 0 & ... & 0 & I_p \end{bmatrix}, \quad D(s) = \frac{N_{q_2}(s)}{d_{q_2}(s)} \end{array} \tag{28b}$$

are a realization of (27b).

Let us define

$$\begin{array}{l} T_\infty = \lim\limits_{s,z\to\infty} T(s,z) = \lim\limits_{s,z\to\infty} \frac{N_{q_2}(s)}{d_{q_2}(z)} = \lim\limits_{s,z\to\infty} \frac{N_{q_1}(s)}{d_{q_1}(z)} \\ d_{k\infty} = -\lim\limits_{z\to\infty} d_k(z), \quad \bar{d}_{k\infty} = \lim\limits_{z\to\infty} (z-c)[d_k(z) - d_{k\infty}], \\ \quad N_{k\infty} = \lim\limits_{z\to\infty} N_k(z), \quad k = 0, 1, ..., q_1 - 1 \\ d_{l\infty} = -\lim\limits_{a\to\infty} d_l(s), \quad \bar{d}_{l\infty} = \lim\limits_{s\to\infty} s^\alpha[d_l(s) - d_{l\infty}], \\ \quad N_{l\infty} = \lim\limits_{s\to\infty} N_l(s), \quad l = 0, 1, ..., q_2 - 1. \end{array} \tag{29}$$

THEOREM 3. *There exists a positive realization of $T(s,z)$ given by (28), if*

$$T_{\infty} \in R_{+}^{p\times m} \tag{30a}$$

$$d_{k\infty} \in R_{+} \;\; for \;\; k=0,1,...,q_1-2 \;\; and \;\; d_{q_1-1,\infty} \;\; is\ arbitrary \tag{30b}$$

$$N_{k\infty} \in R_{+}^{p\times m} \;\; for \;\; k=0,1,...,q_1-1 \tag{30c}$$

$$d_{l\infty} \in R_{+}, \; N_{l\infty} \in R_{+}^{p\times m} \;\; for \quad l=0,1,...,q_2-1 \tag{30d}$$

$$\bar{d}_{k\infty} \in R_{+} \;\; for \;\; k=0,1,...,q_1-1 \;\; and \;\; \bar{d}_{l\infty} \in R_{+} \;\; for \;\; l=0,1,...,q_2-1. \tag{30e}$$

P r o o f. If $T_{\infty} \in R_{+}^{p\times m}$, then $D \in R_{+}^{p\times m}$ since $T_{\infty} = \lim_{s,z\to\infty} T(s,z)$. From (24b) and the form of $A(z)$ it follows that if the condition (30b) is met, then A_{11} is a Metzler matrix, i.e. $A_{11} \in M_{n_1}$. If the condition (30c) is satisfied, then by (24b) $B_1 \in R_{+}^{n_1\times m}$. Similarly, if (30d) holds, then by (25a) we have $A_{22} \in R_{+}^{n_2\times n_2}$ and $B_2 \in R_{+}^{n_2\times m}$. From (25) it follows that $A_{12} \in R_{+}^{n_1\times n_2}$ and $A_{21} \in R_{+}^{n_2\times n_1}$, if the condition (30e) is satisfied. ■

If the assumptions of Theorem 2 are satisfied, then a positive fractional realization (14) of (27) can be found by using the following procedure.

PROCEDURE.

Step 1. Write $T(s,z)$ in the forms (27).
Step 2. Knowing $d_k(z)$, $N_k(z)$ for $k=0,1,...,q_1-1$ and $d_l(s)$, $N_l(s)$ for $l=0,1,...,q_2-1$ and using one of the well-known methods find realizations $A(z)$, $B(z)$, $C(z)$, $D(z)$ and $A(s)$, $B(s)$, $C(s)$, $D(s)$, for example - the realizations (28).
Step 3. Using (24) find matrices $A_{11}, A_{22}, B_1, B_2, C_1, C_2$ and D.
Step 4. Using (25a) (or (25b)) find matrices A_{12} and A_{21}.

5. Example

Find a positive fractional realization (14) of the transfer matrix

$$T(s,z) = \frac{1}{(s^{\alpha}-1)((z-c)^2-2(z-c)-1)} \times \tag{31}$$

$$\begin{bmatrix} s^{\alpha}((z-c)^2+2(z-c)+3)-4(z-c)-4(s^{\alpha}-1)((z-c)^2+2(z-c)+2) \\ (s^{\alpha}-1)((z-c)^2+4(z-c)+1)(s^{\alpha}-1)((z-c)^2+3(z-c)+2) \end{bmatrix}.$$

In this case $m=p=2$, $q_1=1$ and $q_2=2$.

Using Procedure, we obtain the following.

Step 1. The transfer matrix in the form (27a) is

$$T(s,z) = \frac{1}{(z-c)^2 - 2(z-c) - 1} \times \tag{32a}$$

$$\begin{bmatrix} (z-c)^2 + 2(z-c) + 3 & (z-c)^2 + 2(z-c) + 2 \\ (z-c)^2 + 4(z-c) + 1 & (z-c)^2 + 3(z-c) + 2 \end{bmatrix}$$

$$+\frac{1}{s^\alpha - 1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

where

$$d_1(z) = (z-c)^2 - 2(z-c) - 1,$$

$$N_1(z) = \begin{bmatrix} (z-c)^2 + 2(z-c) + 3 & (z-c)^2 + 2(z-c) + 2 \\ (z-c)^2 + 4(z-c) + 1 & (z-c)^2 + 3(z-c) + 2 \end{bmatrix},$$

$$d_0(z) = -1, \quad N_0(z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and in the form (27b) is

$$T(s,z) = \frac{1}{s^\alpha - 1}\begin{bmatrix} s^\alpha & s^\alpha - 1 \\ s^\alpha - 1 & s^\alpha - 1 \end{bmatrix}$$
$$+\frac{1}{(z-c)^2 - 2(z-c) - 1}\begin{bmatrix} \frac{s^\alpha(4(z-c)+4) - 4(z-c) - 4}{s^\alpha - 1} & 4(z-c) + 3 \\ 6(z-c) + 2 & 5(z-c) + 3 \end{bmatrix} \tag{32b}$$

where

$$d_2(s) = s^\alpha - 1, \quad N_2(s) = \begin{bmatrix} s^\alpha & s^\alpha - 1 \\ s^\alpha - 1 & s^\alpha - 1 \end{bmatrix},$$

$$d_1(s) = -2, \quad N_1(s) = \begin{bmatrix} \frac{4s^\alpha - 1}{s^\alpha - 1} & 4 \\ 6 & 5 \end{bmatrix},$$

$$d_0(s) = -1, \quad N_0(s) = \begin{bmatrix} \frac{4s^\alpha - 1}{s^\alpha - 1} & 3 \\ 2 & 3 \end{bmatrix}.$$

Step 2. Using (28) we obtain

$$A(z) = [-d_0(z)I_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B(z) = [N_0(z)] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D(z) = \frac{N_1(z)}{d_1(z)} = \frac{1}{(z-c)^2 - 2(z-c) - 1} \tag{33a}$$
$$\times \begin{bmatrix} (z-c)^2 + 2(z-c) + 3 & (z-c)^2 + 2(z-c) + 2 \\ (z-c)^2 + 4(z-c) + 1 & (z-c)^2 + 3(z-c) + 2 \end{bmatrix}$$

and

$$A(s) = \begin{bmatrix} 0 & -d_0(s)I_2 \\ I_2 & -d_1(s)I_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix},$$

$$B(s) = \begin{bmatrix} N_0(s) \\ N_1(s) \end{bmatrix} = \begin{bmatrix} \frac{4s^\alpha-4}{s^\alpha-1} & 3 \\ 2 & 3 \\ \frac{4s^\alpha-4}{s^\alpha-1} & 4 \\ 6 & 5 \end{bmatrix}, \tag{33b}$$

$$C(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D(s) = \frac{N_2(s)}{d_2(s)} = \frac{1}{s^\alpha - 1} \begin{bmatrix} s^\alpha & s^\alpha - 1 \\ s^\alpha - 1 & s^\alpha - 1 \end{bmatrix}.$$

Step 3. From (24) and (33) we have

$$\begin{aligned} &A_{11} = \lim_{z\to\infty} A(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \lim_{z\to\infty} B(z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ &C_1 = \lim_{z\to\infty} C(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \lim_{z\to\infty} D(z) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \end{aligned} \tag{34a}$$

and

$$\begin{aligned} &A_{22} = \lim_{s\to\infty} A(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}, \quad B_2 = \lim_{s\to\infty} B(s) = \begin{bmatrix} 4 & 3 \\ 2 & 3 \\ 4 & 4 \\ 6 & 5 \end{bmatrix}, \\ &C_2 = \lim_{s\to\infty} C(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{34b}$$

Step 4. Using (25) and (34) we obtain

$$\begin{aligned} &\lim_{z\to\infty} (z-c) \begin{bmatrix} A(z) - A_{11} & B(z) - B_1 \\ C(z) - C_1 & D(z) - D \end{bmatrix} = \lim_{z\to\infty} \frac{(z-c)}{(z-c)^2 - 2(z-c) - 1} \\ &\times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4(z-c)+4 & 4(z-c)+3 \\ 0 & 0 & 6(z-c)+2 & 5(z-c)+3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 6 & 5 \end{bmatrix} \\ &= \begin{bmatrix} A_{12} \\ C_2 \end{bmatrix} [A_{21} \ \ B_2] = \begin{bmatrix} A_{12} \\ \cdots\cdots\cdots \\ 0 \ 0 \ 1 \ 0 \\ 0 \ 0 \ 0 \ 1 \end{bmatrix} \left[\begin{array}{c|cc} & 4 & 3 \\ A_{21} & 2 & 3 \\ & 4 & 4 \\ & 6 & 5 \end{array} \right] \end{aligned} \tag{35a}$$

and

$$\lim_{s\to\infty} s^{\alpha} \begin{bmatrix} A(s)-A_{22} & B(s)-B_2 \\ C(s)-C_2 & D(s)-D \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{35b}$$

$$= \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix} [A_{12} \;\; B_1] = \begin{bmatrix} A_{21} \\ \cdots\cdots \\ 1 \;\; 0 \\ 0 \;\; 1 \end{bmatrix} \begin{bmatrix} A_{12} & \vdots & \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \end{bmatrix}.$$

From (35) we have

$$A_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{36}$$

The desired realization of (31) is given by (34) and (36).

6. Concluding remarks

A method for computation of a positive fractional realization of a given proper transfer matrix of 2D hybrid systems has been proposed. Sufficient conditions for the existence of a positive realization of a given proper transfer matrix have been established. Note that there exists a positive fractional realization (14) only if the given proper transfer matrix has the special form (27). A procedure for computation of a positive fractional realization has been proposed. The effectiveness of the procedure has been illustrated by a numerical example. An open problem is the formulation of the necessary and sufficient conditions for the existence of solution of the positive realization problem for 2D hybrid systems in the general case. Extensions of those considerations for 2D hybrid systems described by models with structures similar to the 2D general model [18], or the 2D second Fornasini-Marchesini model [3], are also open problems.

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